BOUNDARY VALUES OF ANALYTIC FUNCTIONS

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DONALD EUGENE KIBBEY

A.B., University of Illinois, 1935

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ABSTRACT OF A THESIS

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY IN MATHEMATICS IN THE GRADUATE SCHOOL OF THE UNIVERSITY OF ILLINOIS, 1941

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INTRODUCTION

Bloch has proved that there is an absolute constant $B > 0$ with the property that if $f(z)$ is analytic for $|z| < 1$ and if $|f'(0)| \geq 1$, the range of $f(z)$ contains the interior of some circle of radius B. Doob[2] has proved a theorem of the same type about a class of functions satisfying the conditions of the following definition:

Definition. Let $f(z)$ be analytic for $|z| < 1$ and let A be an open arc on $|z| = 1$. Suppose that $f(0) = 0$ and that

$$\liminf_{n \to \infty} |f(z_n)| \geq 1 \tag{0.1}$$

for any sequence of points $\{z_n\}$ of $|z| < 1$ which converges to a point of A. Then if $mA \geq \rho > 0$, $f(z)$ will be said to have the property $K(\rho)$.

Doob's theorem states that there is a positive function $k(\rho)$ depending only on ρ such that if $f(z)$ has the property $K(\rho)$, its range covers the interior of some circle of radius $k(\rho)$.

In this paper we shall consider functions with the property $K(\rho)$ that are bounded, and the purpose is to obtain an explicit evaluation for a positive function $k(\rho, M)$ depending only on ρ and M such that if $f(z)$ has the property $K(\rho)$ and if $|f(z)| < M$ for $|z| < 1$, then the range of $f(z)$ covers the interior of some circle of radius $k(\rho, M)$. This purpose is achieved in Theorems 7 and 8. If the quantitative results of this paper are extended to the class of unbounded functions with the property $K(\rho)$, another evaluation of Bloch's constant can be obtained.[3]

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1). E. Landau, Mathematische Zeitschrift, vol. 30, 1929, pp. 622-623, showed that .43 is a possible value for B.

2). J. L. Doob, Annals of Mathematics, vol. 36, 1935, p. 120.

3). J. L. Doob, Annals of Mathematics, vol. 36, 1935, p. 120 and pp. 124-126.

I. Preliminary Discussion

Let $f(z)$ be a bounded function, analytic for $|z| < 1$. Define the functions $f(z)$, $f_\alpha(z)$ as follows:

$$(1.1)\qquad f(z) = \frac{\log f(z) + 1}{\log f(z) - 1}, \qquad f_\alpha(z) = \frac{\log \frac{\alpha - f(z)}{\bar{\alpha} f(z) - 1} + 1}{\log \frac{\alpha - f(z)}{\bar{\alpha} f(z) - 1} - 1}.$$

The functions $f(z)$ and $f_\alpha(z)$ are analytic for $|z| < 1$; also $|f(z)| < 1$, $|f_\alpha(z)| < 1$. We shall postpone the selection of branches of $\log f(z)$ and $\log \frac{\alpha - f(z)}{\bar{\alpha} f(z) - 1}$ until later. In what follows we shall use the symbols $R(w)$ and $I(w)$ for the real and imaginary parts respectively of the complex number w.

By a theorem of Fatou,[4] $\lim_{r \to 1} f(re^{i\theta})$ exists almost everywhere on the interval $0 \le \theta < 2\pi$, defining a boundary function almost everywhere on $|z| = 1$, $z = e^{i\theta}$. We shall use this theorem without further reference when we discuss the boundary values of the functions $f(z)$, $f_\alpha(z)$ and the integrals of these functions on the boundary.

Lemma 1. Let $f(z)$ be a function analytic for $|z| < 1$ and suppose that $|f(z)| < 1$ there. Let A be an arc of length $\rho = 2\pi\theta$ on $|z| = 1$ and suppose that

$$(1.3)\qquad \liminf_{n \to \infty} |f(z_n)| \ge \delta$$

for any sequence $\{z_n\}$ of points converging to a point of A. Suppose $f(z) \not\equiv 0$, α where $0 < |\alpha| < \frac{\delta}{1+\delta}$, $\delta < 1$ and that

$$(1.4)\qquad R(1 - f(0)) = 2\pi\epsilon_1^2.$$

Then

$$(1.5)\qquad \int_A |1 - f_\alpha(z)|\,|dz| \le 2\pi\epsilon_1 \sqrt{2\theta} \cdot \log \frac{\delta e}{\delta - \delta|\alpha| - |\alpha|},$$

where $f(z)$, $f_\alpha(z)$ are as defined in (1.1), with the branches of $\log f(z)$ and $\log \frac{\alpha - f(z)}{\bar{\alpha} f(z) - 1}$ chosen as explained below.

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4). P. Fatou, Acta Mathematica, vol. 30, (1906), pp. 366-367.

The Cauchy integral formula and Schwarz's inequality give the result

(1.8) $$\int_A |1-f(z)|\,d\theta < 2\epsilon_1\sqrt{\rho\pi} = 2\pi\epsilon_1\sqrt{2\sigma}.$$

Suppose that f(z) is continuous on A. Then we can choose that branch of $\log \frac{\alpha - f(z)}{f(z)[\bar{\alpha} f(z)-1]}$ for z on A which is real when $\frac{\alpha - f(z)}{f(z)[\bar{\alpha} f(z)-1]}$ is real. This determines $\log \frac{\alpha - f(z)}{f(z)[\bar{\alpha} f(z)-1]}$ uniquely for $|z|<1$ also. Let us choose that branch of $\log \frac{\alpha - f(z)}{\bar{\alpha} f(z)-1}$ for which the relation

(1.14) $$-\pi < I\left(\log \frac{\alpha - f(0)}{\bar{\alpha} f(0)-1}\right) \le \pi$$

is true and that branch of log f(z) for which the equation

(1.15) $$\log \frac{\alpha - f(z)}{f(z)[\bar{\alpha} f(z)-1]} = \log \frac{\alpha - f(z)}{\bar{\alpha} f(z)-1} - \log f(z)$$

holds. By maximizing the function $|f(z) - f_\alpha(z)|$ for z on the arc A, we proceed directly to the conclusion of the lemma.

We have assumed that f(z) is continuous on A. If this is not so, consider the function $f_1(z) = f(tz)$ for t a positive number < 1. The function $f_1(z)$ is analytic for $|z| \le 1$ and an analogue of Lemma 1 for $f_1(z)$ holds. When t approaches 1 we obtain Lemma 1 for f(z).

Lemma 2. Let f(z) be a function analytic for $|z|<1$ and such that $|f(z)|<1$ there. Let $\int_A |f(z)|\,d\theta \le \epsilon$, where A is an arc of length $\rho = 2\pi\sigma > 0$ on $|z|=1$. Then

(2.1) $$|f(0)| \le \left(\frac{\epsilon}{2\pi\sigma}\right)^{\sigma}.$$

Define $u(\theta)$ on $|z|=1, z = re^{i\theta} = x+iy$, to have the value $\frac{K}{2\pi\sigma}$ on A and the value $\frac{K}{2\pi(\sigma-1)}$ on CA, where K is an arbitrary constant. Using the Poisson integral formula, we define

(2.2) $$u(r,\phi) = \frac{1}{2\pi}\int_0^{2\pi} u(\theta)\frac{1-r^2}{1-2r\cos(\theta-\phi)+r^2}\,d\theta$$

in $|z|<1$ as a harmonic function of x and y. Define

(2.4) $$V(x,y) = \int_{-}^{x,y} \frac{\partial u}{\partial y}\,dx + \frac{\partial u}{\partial x}\,dy.$$

and define $g(z) = e^{u+iv}$. Then

(2.6) $$f(0) = f(0)\cdot g(0) = \frac{1}{2\pi}\int_0^{2\pi} f(z)\cdot g(z)\,d\theta,$$

and

$$|f(0)| \le \frac{\epsilon}{2\pi} e^{\frac{K}{2\pi\sigma}} + (1-\sigma) e^{\frac{K}{2\pi(\sigma-1)}}.$$

Minimizing the function on the right-hand side of the inequality gives the conclusion of Lemma 2.

Lemma 3. Let $f(z)$ be a function analytic for $|z|<1$ and suppose that $|f(z)|<1$ there. Let A be an arc of length $\rho = 2\pi\sigma$ on $|z|=1$ and suppose that

(3.1) $$\liminf_{n\to\infty} |f(z_n)| \ge \delta$$

for any sequence $\{z_n\}$ of points of $|z|<1$ converging to a point of A. Suppose that $f(z) \neq 0, \alpha$ and that $0 < |\alpha| < \frac{\delta}{1+\delta}$, $\delta < 1$. Let ϵ_2 be defined by

(3.2) $$\epsilon_2 = \left[\epsilon_1 \sqrt{\frac{2}{\sigma}} \cdot \log \frac{\delta e}{\delta - \delta|\alpha| - |\alpha|}\right]^{\sigma},$$

where $2\pi\epsilon_1^2 = R[1-f(0)]$.

Then

(3.3) $$\left|\log \frac{\alpha - f(0)}{\bar{\alpha} f(0) - 1}\right| > \frac{2}{\epsilon_2},$$

where the branch of $\log \frac{\alpha - f(z)}{\bar{\alpha} f(z) - 1}$ is chosen as in (1.14).

The proof follows from the application of Lemma 2 to the function $1 - f_\alpha(z)$.

Theorem 4. Let $f(z)$ be a function analytic for $|z|<1$ and suppose that $|f(z)|<1$ there. Let A be an arc of length ρ on $|z|=1$, and suppose that

(4.1) $$\liminf_{n\to\infty} |f(z_n)| \ge \delta$$

for any sequence $\{z_n\}$ of points of $|z|<1$ converging to a point of A. Suppose that $f(z) \neq 0, \alpha$ and that $0 < |\alpha| < \frac{\delta}{1+\delta}$, $\delta < 1$. Let ϵ_0 be a positive number such that

(4.2) $$|f(0)| < \epsilon_0,$$

and define

(4.3) $$\epsilon_3 = (1+\epsilon_0) e^{1 - \frac{2}{\epsilon_2^2}},$$

where

$$\epsilon_2' = \left[\epsilon_1' \sqrt{\frac{2}{\sigma}} \cdot \log \frac{\delta e}{\delta - \delta|\alpha| - |\alpha|} \right]^{\sigma}, \tag{4.4}$$

$$2\pi \epsilon_1'^2 = R\left[\frac{-2}{\log c f(0) - 1} \right], \tag{4.5}$$

and c is a complex number such that $|c| = 1$ and such that $c \cdot \frac{\alpha - f(0)}{\bar{\alpha} f(0) - 1}$ is real and positive. Then

$$|\alpha - f(0)| < \epsilon_3 : \tag{4.6}$$

Consider the function $\phi(z) = c \cdot f(z)$. This function satisfies the hypotheses of Lemma 3, and (4.6) follows.

II. FUNCTIONS WITH THE PROPERTY K(ρ).

Theorem 5. Let F(z) be a function with the property K(ρ) such that $|F(z)| < M$ for $|z| < 1$. Let a and b be any two points in $\{|z| < 1\}$ such that $0 < \frac{|a-b|}{M+|a|} < \frac{1-|a|}{M+1}$. Suppose that $F(z) \neq a, b$. Let ϵ_0 be a positive number such that

$$\frac{|a|}{M+|a|} < \epsilon_0 \tag{5.1}$$

and define

$$\epsilon_4 = (1+\epsilon_0) e^{1 - \frac{2}{\epsilon_2''}} \tag{5.2}$$

where

$$\epsilon_2'' = \left[\epsilon_1'' \sqrt{\frac{2}{\sigma}} \cdot \log \frac{e}{1 - \frac{|a-b|}{M+|a|} - \frac{|a-b|}{1-|a|}} \right]^{\sigma}$$

$$2\pi \epsilon_1''^2 = R\left(\frac{-2}{\log c \cdot \frac{a}{M+|a|} - 1} \right).$$

and where c is defined as in Theorem 4. Then it follows that

$$\frac{|b|}{M+|a|} < \epsilon_4. \tag{5.3}$$

Form the function

$$f(z) = \frac{a - F(z)}{M + |a|},$$

and apply Theorem 4 to f(z).

Lemma 6. Let F(z) be a function with the property K($2\pi\sigma$) such that $|F(z)| < M$ for $|z| < 1$. Let r_1, r_2, r_3 be three positive numbers and a and b two points such that

(6.0)
$$1)\quad 0 < |a| < r_1 < r_2 < |b| < r_3 < 1,$$
$$2)\quad (M+1)\, r_3 < M - 2r_1 M - r_1^2,$$
$$3)\quad F(z) \neq a, b.$$

Then it follows that

(6.1)
$$\frac{|b|}{M + r_1} < \epsilon_4',$$

where

$$1 - \frac{2}{\epsilon_2'''}$$

(6.2)
$$\epsilon_4' = (1 + r_1)\, e,$$

(6.3)
$$\epsilon_2''' = \left[\epsilon_1''' \sqrt{\frac{2}{\sigma}} \cdot \log \frac{e\,(M + r_1)(1 - r_1)}{M - 2r_1 M - r_1^2 - (M+1) r_3} \right]^{\sigma}$$

(6.4)
$$2\pi \epsilon_1'''^2 = \frac{2}{1 - \log \frac{r_1}{M + r_1}}.$$

We can take r_1 as the ϵ_0 of Theorem 5 and the result follows.

Theorem 7. Let F(z) be a function with the property K($2\pi\sigma$) such that $|F(z)| < M$ for $|z| < 1$. Let r_1, r_2, r_3 be positive numbers satisfying the relations,

(7.1)
$$r_1 < r_2 < r_3 < \frac{M - 2r_1 M - r_1^2}{M+1}$$

Then if r_1 is decreased, if necessary, to satisfy the condition,

(7.2)
$$\log \frac{e\,(M + r_1)(1 + r_1)}{r_2} \leq 2 \left(\frac{\pi \sigma}{2} \right)^{\frac{\sigma}{2}} \left[\frac{\sqrt{\log \frac{e(M + r_1)}{r_1}}}{\log \frac{e(M + r_1)(1 - r_1)}{M - 2r_1 M - r_1^2 - (M+1) r_3}} \right]^{\sigma},$$

the range of F(z) will include either $\{|w|<\lambda_1\}$ or $\{\lambda_2<|w|<\lambda_3\}$.

Suppose the theorem is not true. Then there is a function F(z) and a triple $\lambda_1,\lambda_2,\lambda_3$ satisfying (7.1) and (7.2) such that either the annulus $\{\lambda_2<|w|<\lambda_3\}$ or the circle $\{|w|<\lambda_1\}$ is not covered; that is, there is a point a in the circle and a point b in the annulus such that $F(z)\neq a,b$. Application of Lemma 6 gives a contradiction.

Theorem 8. Let F(z) be a function with the property K(ρ) such that $|F(z)|<M$ for $|z|<1$. The range of F(z) includes a circle of radius k(ρ, M) depending only on ρ, M, where k(ρ, M) is the minimum of the two numbers λ_1 and $\frac{\lambda_3-\lambda_2}{2}$, and where the numbers $\lambda_1,\lambda_2,\lambda_3$ are as in Theorem 7.

VITA

Donald Eugene Kibbey was born February 19, 1912, in Junction City, Kansas, where he received his elementary and much of his secondary school education. He was graduated from Westport High School, Kansas City, Missouri, in 1929, and from the Kansas City Junior College in 1931. He matriculated at the University of Illinois in September, 1933, and received the degrees of Bachelor of Arts in 1935 and Master of Arts in 1936 from that institution. He was an assistant in mathematics at the University of Illinois from September, 1936, to September, 1939. Since September, 1939, he has been in the department of mathematics at the University of Kansas City.